Differential Geometry and its Applications 27 (2009) 15–22

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Rectification of spheres of co-dimension 1 and 2 in $\mathbb{R}^n$

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ARTICLE INFO

Article history:
Received 5 October 2006
Received in revised form 20 August 2007
Available online 18 October 2008
Communicated by O. Kowalski

MSC:
primary 53A04
secondary 53B20

Keywords:
Rectification
Rectifiable pencil of spheres
Rectifiable family of circles
Diffeomorphism
Conformal mapping
Nomography

ABSTRACT

Consider a pencil $\mathbb{S}$ of k-dimensional surfaces in $\mathbb{R}^n$ passing through the origin. A rectification of $\mathbb{S}$ is a germ Φ of a diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that the image $\Phi(s)$ of each surface $s \in \mathbb{S}$ belongs to an affine k-subspace. Here $\Phi(s)$ denotes, more precisely, the restriction of Φ to a germ of such a surface s. The main result of the paper is the following. Let $\mathbb{S}$ be a rectifiable pencil of spheres in $\mathbb{R}^n$ of co-dimension 1 or 2. Assume that $\mathbb{S}$ is large enough and that the tangent planes to spheres in $\mathbb{S}$ are in general position. Then all spheres in $\mathbb{S}$ have a common point different from the origin.

0. Introduction

In 1970s, A.G. Khovanskii proved the following theorem motivated by some problems in nomography [3]: if a rectifiable pencil $\mathbb{S}$ of circles in $\mathbb{R}^2$ passing through the origin and contains at least 7 circles, then all circles in $\mathbb{S}$ have a common point different from the origin [2]. The author extended this result to circles in $\mathbb{R}^3$ [1]: for the same conclusion, one needs sufficiently many circles whose tangent lines at the origin are generic (do not belong to the same algebraic hypersurface of small degree). The present paper is a further development of this result. Consider a pencil $\mathbb{S}$ of k-dimensional surfaces in $\mathbb{R}^n$ passing through the origin. A rectification of $\mathbb{S}$ is a germ Φ of a diffeomorphism $(\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that the image $\Phi(s)$ of each surface $s \in \mathbb{S}$ belongs to an affine k-subspace. Here $\Phi(s)$ denotes, more precisely, the restriction of Φ to a germ of such a surface s. In this case, we call $\mathbb{S}$ a rectifiable pencil of surfaces. The main result of the paper is the following. Let $\mathbb{S}$ be a rectifiable pencil of spheres in $\mathbb{R}^n$ of co-dimension 1 or 2. Assume that $\mathbb{S}$ is large enough and that the tangent planes to spheres in $\mathbb{S}$ are in general position. Then all spheres in $\mathbb{S}$ have a common point different from the origin. The paper explains that the co-dimension of the spheres is what matters for the problem, not the dimension. V. Timorin discovered that similar statements fail for circles in higher dimensions. For example in dimension 4, the quaternionic Hopf fibrations can be used to generate rectifiable families of circles with a different geometry [4,5]. Complex, quaternionic, octonionic structures, representation of Clifford algebras, and, more generally, normed pairings provide interesting examples of rectifiable families of circles [6,7].

We summarize the following definitions and notation from the introduction.

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2008.06.003

Definitions and notation 1 *(Rectifiable pencil of surfaces).* A pencile $\mathbb{S}$ of k-dimensional surfaces in $\mathbb{R}^n$ passing through the origin is said to be *locally rectifiable* if there is a *germ of diffeomorphism* $\Phi:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ such that the image $\Phi(s)$ of each surface $s\in\mathbb{S}$ belongs to an affine k-subspace. We call the point 0 the *center* of the pencil and the pencil itself is called *central*. A pencil is called *simple*, if different surfaces in the pencil have different tangent planes.

Remark 2. If a pencil of surfaces is locally rectifiable near the center of the pencil, then it is simple. We are interested in the behavior of the surfaces in a rectifiable pencil near the center.

Consider a family of differentiable functions $y_\alpha:\mathbb{R}^n\to\mathbb{R}$ depending on the vector $\alpha=(\alpha_1,\dots,\alpha_n)$, where α varies on all points of $\mathbb{R}^n$. Suppose that $y_\alpha(0)=0$ and $\nabla y_\alpha(0)=\alpha$ for all values of α in $\mathbb{R}^n$. The graphs of the C^m functions y_α form a pencil $\mathbb{S}$ of hypersurfaces centered at the origin.

Bearing this notation in mind, we have the following proposition generalizing a proposition from [1] and [2].

Proposition 3. *Suppose that a simple pencil $\mathbb{S}$ of hypersurfaces corresponding to the functions y_α subject to the conditions $y_\alpha=0$ and $\nabla y_\alpha(0)=\alpha$ is locally rectifiable near the origin by means of a C^m diffeomorphism. Then for every i, $1\leqslant i\leqslant m$, there exist n-variable polynomials P^j_{2i-1}, $(j=1,\dots,n)$ of degree at most $(2i-1)$ such that*

$$\frac{\partial^i y_\alpha}{\partial x_j^i}(0)=P^j_{2i-1}(\alpha),\quad 1\leqslant j\leqslant n,\ 1\leqslant i\leqslant m.$$

For the proof, we need a simple lemma which is an immediate consequence of the implicit differentiation.

Lemma 4. *Suppose that there exists a C^m function $G:\mathbb{R}^{n+1}\to R$ such that*

$$G\bigl(x_1,x_2,\dots,x_n,y_\alpha(x)\bigr)=0,$$
$$\frac{\partial G(x_1,x_2,\dots,x_n,y)}{\partial y}(0)=1.$$

Let

$$G(x_1,\dots,x_n,y_\alpha)=\sum a_{p_1,\dots,p_{n+1}}x_1^{p_1}\cdots y_\alpha^{p_{n+1}},$$
$$y_\alpha(x)=\sum b_{p_1,\dots,p_n}x_1^{p_1}\cdots x_n^{p_n}$$

be the Taylor expansions at the origin for the functions G and y_α respectively. Then the coefficients $b_{p_1,\dots,p_n}$, are equal to some n-variable polynomials of degree $(2i-1)$ in the coefficients $a_{p_1\cdots p_{n+1}}$.

Proof of Lemma 4. First of all, since $y_\alpha(0)=0$, we have $G(0)=0$. By the implicit function theorem, the equation

$$G\bigl(x,y_\alpha(x)\bigr)=0$$

can be solved locally for the function $y_\alpha(x)$, and this function is a smooth function of class C^m, in fact we have

$$\frac{\partial G}{\partial x_k}+\frac{\partial G}{\partial y}\frac{\partial y}{\partial x_k}=0. \tag{$*$}$$

Note that we have used the symbol y instead of y_α and we will do this in all formulas where there is no risk of confusion at the rest of our discussion.

Solving this equations for y_{x_k} at the origin we get

$$\frac{\partial y}{\partial x_k}(0)=-\frac{\partial G}{\partial x_k}(0),$$

noting that $\frac{\partial G}{\partial y}(0)=1$.

This is equivalent to

$$b_{0\dots0k\dots0}=-a_{0\dots0k\dots0}.$$

Similarly by differentiating Eq. ($*$) with respect to x_j, we get

$$G_{x_kx_j}+G_{x_ky}y_{x_j}+y_{x_k}G_{yx_j}+G_{yy}y_{x_j}y_{x_k}+G_yy_{x_kx_j}=0,\quad 1\leqslant k,j\leqslant n.$$

Evaluating this equation at the origin and solving the resulting expression with respect to

$$\frac{\partial^2 y}{\partial x_k\partial x_j}(0),$$

we obtain

$$y_{x_k x_j}(0) = -G_{x_k x_j}(0) - G_{x_k y}(0) y_{x_j}(0) - y_{x_k}(0) G_{y x_j}(0) - G_{yy}(0) y_{x_j}(0) y_{x_k}(0).$$

By substituting the values of $y_{x_k}(0)$ and $y_{x_j}(0)$ from previous expressions and the values of the second partial derivatives of G appearing in the above expression, we can easily obtain the expression for

$$\frac{\partial^2 y}{\partial x_k \partial x_j}(0).$$

Since the term $G_{yy} y_{x_k} y_{x_k}(0)$ is a polynomial of degree 3 in the coefficients of G, we see that

$$\frac{\partial^2 y}{\partial x_k \partial x_j}(0)$$

is a polynomial of degree 3. This proves the lemma for $i = 2$.

Continuing differentiation and evaluating the result at 0, then solving with respect to the third partial derivative of y, and finally substituting all the other expressions from previous ones we can obtain the formula for $i = 3$. The higher expressions can be consecutively obtained in a similar way. □

Proof of Proposition 3. Consider a diffeomorphism Φ rectifying the pencil $\mathbb{S}$ of hypersurfaces. Let T be an arbitrary non-singular linear mapping of the space. The diffeomorphism $T \circ \Phi$ also rectifies the pencil $F_\alpha(x)$. By a suitable choice of T, we can arrange that the rectifying diffeomorphism has the identity differential at the point 0. The rectifying diffeomorphism is now given by

$$T \circ \Phi(x) = \left(f^1(x), \ldots, f^{n+1}(x)\right)$$

where $f^i(x) = x_i + \text{higher order terms}, 1 \leqslant i \leqslant n$.

In the space $\mathbb{R}^{n+1}$ having the coordinates $(u_1, \ldots, u_{n+1})$, the pencil of surfaces $F_\alpha(x)$ is given by the equation

$$\langle \beta, u \rangle = 0,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the parameter vector, $\beta = (\alpha, 1)$, and $u = (u_1, u_2, \ldots, u_{n+1})$.

Consequently, the surfaces $F_\alpha(x)$ are given by the equations $G_\alpha(x) = 0$, where

$$G_\alpha(x) = \alpha_1 f^1(x) + \cdots + \alpha_n f^n(x) + f^{n+1}(x).$$

Now the coefficients $a_{p_1 \ldots p_{n+1}}$, in the Taylor expansion of the surfaces G_α depend linearly on $\alpha_1, \ldots, \alpha_n$. We set

$$P^j_{2i-1}(\alpha) = \frac{\partial^i y}{\partial x^i_j}(0), \quad 1 \leqslant i \leqslant n, \ 1 < i \leqslant m.$$

Since by Lemma 4

$$\frac{\partial^i y}{\partial x^i_j}(0)$$

is a polynomial of degree at most $(2i - 1)$ in the coefficients of Taylor expansion of G and these coefficients are in turn depend linearly on $\alpha_1, \ldots, \alpha_n$, the expressions

$$P^j_{2i-1}(\alpha)$$

are in fact polynomials of degree at most $(2i - 1)$ in $\alpha_1, \ldots, \alpha_n$. The proof of Proposition 3 is now complete. □

Having proved the above proposition, we are now ready to state the fundamental theorem of rectification for the spheres of co-dimension 1 in $\mathbb{R}^{n+1}$.

Theorem 5. *Consider a pencil $\mathbb{S}$ of spheres in $\mathbb{R}^{n+1}$ of co-dimension 1 passing through the origin. Then there exists a germ of diffeomorphism $\Phi : (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^{n+1}, 0)$ such that the image $\Phi(s)$ of each $s \in \mathbb{S}$ belongs to an affine n-subspace if and only if all the spheres in $\mathbb{S}$ pass through one point different from the origin.*

First proof. The equations for a pencil of spheres with center at 0 can be easily reduced to the form $\langle \beta, x \rangle = A\langle x, x \rangle$, where $\beta = (\alpha_1, \ldots, \alpha_n, 1)$, $x = (x_1, \ldots, x_{n+1})$, and $A = A(\alpha)$ is some rational function of the parameters $\alpha_1, \ldots, \alpha_n$. As the proof for a rectifiable bundle of circles in 3-dimensional space $\mathbb{R}^3$ [1], we show that the rectifiability of this pencil is equivalent to the fact that A is a polynomial function of degree 1 in the parameters $\alpha_1, \ldots, \alpha_n$. To do this we consider the last variable

x_{n+1} as an implicit function of the remaining variables. By differentiating this variable with respect to any other variable, say x_i, we get

$$\alpha_i + \frac{\partial x_{n+1}}{\partial x_i} = 2A\left(x_i + x_{n+1}\frac{\partial x_{n+1}}{\partial x_i}\right). \tag{1}$$

Solving this equation at 0 yields

$$\frac{\partial x_{n+1}}{\partial x_i}(0) = \alpha_i. \tag{2}$$

Next, we differentiate Eq. (1) with respect to the same variable x_i to get

$$\frac{\partial^2 x_{n+1}}{\partial x_i^2} = 2A\left(1 + \left(\frac{\partial x_{n+1}}{\partial x_i}\right)^2 + x_{n+1}\frac{\partial^2 x_{n+1}}{\partial x_i^2}\right). \tag{3}$$

By evaluating Eq. (3) at 0 and using the expression (2) we obtain the following formulas.

$$\frac{\partial^2 x_{n+1}}{\partial x_i^2}(0) = 2A\left(1+\alpha_i^2\right). \tag{4}$$

Differentiating Eq. (3) with respect to x_i again and evaluating the resulting expression at 0 and using the formulas from (2) and (4) yields

$$\frac{\partial^3 x_{n+1}}{\partial x_i^3}(0) = -12\alpha_i A^2\left(1+\alpha_i^2\right). \tag{5}$$

Eliminating $2A$ between (4) and (5) yields

$$\left(1+\alpha_i^2\right)\frac{\partial^3 x_{n+1}}{\partial x_i^3}(0) = -3\alpha_i\left[\frac{\partial^2 x_{n+1}}{\partial x_i^2}(0)\right]^2. \tag{6}$$

By Proposition 3 the functions

$$\frac{\partial^3 x_{n+1}}{\partial x_i^3}(0), \qquad \frac{\partial^2 x_{n+1}}{\partial x_i^2}(0)$$

are n-variable polynomial functions of degree at most 5 and 3 in $\alpha_1, \ldots, \alpha_n$ respectively. Eq. (6) holds for all values of $\alpha_1, \ldots, \alpha_n$, corresponding to spheres in the pencil, i.e., at least for a N-tuple of spheres in general position. Since n-variable polynomials of degree d which coincide at a N-tuple of spheres in general position coincide identically, Eq. (6) is in fact an identity. This implies that $(1+\alpha_i^2)$ divides

$$\frac{\partial^2 x_n}{\partial x_i^2}(0).$$

as a polynomial in α.

Since the latter is a polynomial of degree at most three, it follows that $A(\alpha)$ is a polynomial function of degree 1 in $\beta = (\alpha_1, \ldots, \alpha_n, 1)$. i.e., $A(\alpha) = \langle c, \beta\rangle$, where $c = (c_1, \ldots, c_{n+1})$ is a constant vector in $\mathbb{R}^{n+1}$. Thus the equations for a rectifiable pencil of spheres have the form:

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + x_{n+1} = (c_1\alpha_1 + \cdots + c_n\alpha_n + c_{n+1})\langle x, x\rangle.$$

Since $\alpha_1, \ldots, \alpha_n$ are arbitrary parameters, it is enough to choose α from the set $\{0, e_1, \ldots, e_n\}$, where e_i are standard basis vectors of $\mathbb{R}^n$. It follows that: $x_i = c_i\langle x, x\rangle$, $i = 1, \ldots, n+1$. We see that all the spheres in the pencil pass through the single point

$$\left(\frac{c}{\langle c, c\rangle}\right),$$

different from the origin.

Conversely, if the pencil passes through one point different from the center of the pencil, then by making an inversion with respect to a sphere centered at the this point, we can easily map the pencil into affine spaces. □

Second proof. By the first proof, it is enough to prove the theorem only in one direction. This proof is independent of the first proof and is given by induction on n for a pencil containing at least $n+5$ spheres in general position in $\mathbb{R}^n$. Having said that, we can now restate our theorem as: suppose that a pencil of spheres passing through the origin and containing at least $n+5$ $(2 \leqslant n)$, spheres in general position in $\mathbb{R}^n$ is locally rectifiable, then all the spheres pass through one point different from the origin. For $n=2$, this is just the rectification theorem of the bundle of circles in $\mathbb{R}^2$ [2]. For $n=3$ we have at least eight 2-dimensional spheres all of which lie in $\mathbb{R}^3$. We fix one sphere and call it S_0. Then the intersections of S_0 with the remaining spheres is a bundle of circles, say C_0. Let ψ be the stereographic projection which sends S_0 into a plane. If T is the desired rectifying map, then the composition map $T \circ \psi^{-1}$ is a map which sends a bundle of circles in a plane – the image of C_0 under the projection ψ – into the straight lines. Since this bundle contains at least 7 circles, by the result for the rectifiable bundle of circles in $\mathbb{R}^2$ [2], it should pass through a second point p different from the point $\psi(0)$. It follows that all the circles in the bundle C_0 and consequently all the spheres in the 3-dimensional space having at least eight 2-dimensional spheres pass through $\psi^{-1}(p)$ which is different from the origin.

Now suppose that the statement is true for $n = k$. We will prove that it is true for $n = k + 1$. So suppose that a pencil of k-dimensional spheres passing through the origin and containing at least $(k + 1) + 5$ spheres in general position in $\mathbb{R}^{k+1}$ is locally rectifiable. As above we fix one sphere and take the intersections of that with the remaining ones. Clearly these intersections give rise to a pencil of $(k - 1)$-dimensional spheres passing through the origin and containing at least $(k + 5)$ spheres. By induction, the new pencil passes through another point distinct from the origin. It follows that the main pencil also passes through this second point. The proof is complete. □

Next theorem shows that the same result holds for the spheres of co-dimension 2 in $\mathbb{R}^n$. However there is a little difference between this and the previous case. Since there is a possibility of just single-point intersection between the 2-dimensional spheres of 4-dimensional space $\mathbb{R}^4$ at the origin, we will discuss this case separately and will apply the induction proof for $n = n_0 + 4$, where n_0 is a positive integer.

Theorem 6. *Consider a pencil $\mathbb{S}$ of spheres in $\mathbb{R}^n$ of co-dimension 2 passing through the origin. Then there exists a germ of diffeomorphism $\Phi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that the image $\Phi(s)$ of each $s \in \mathbb{S}$ belongs to an affine $(n-2)$-subspace if and only if all the spheres in $\mathbb{S}$ pass through one point different from the origin.*

Proof. First suppose that $n = 4$. In this case the pencil consists of 2-dimensional spheres. The system of equation defining the simple pencil of spheres with center at $(0, 0, 0, 0)$ has the following form:

$$\begin{cases} z = kx + my + A(x^2 + y^2 + z^2 + t^2), \\ t = nx + ly + B(x^2 + y^2 + z^2 + t^2) \end{cases} \tag{0.1}$$

and, $A = A(k, m, n, l)$ and $B = B(k, m, n, l)$ are some rational functions of the parameters k, m, n, and l. We show that the rectifiability of the pencil is equivalent to the fact that the functions A and B are not only polynomial functions of degree 1 in the parameters k, m, n, l but also their constant coefficients satisfy some symmetric relations. Using argument relatively similar to the argument for the bundle of circles in $\mathbb{R}^3$ [1], for the partial derivatives of the functions z, t with respect to the variables x, y we can easily show that A, B are polynomial functions of degree 1 in the parameters k, m, n, l. To do this we solve the equations for the spheres in the pencil up to terms of fourth order of smallness, so by writing down the Taylor expansions of both functions $z(x, y)$ and $t(x, y)$ with respect to x and y, for the Taylor series we obtain

$$z(x, y) = kx + my + \sum_{i,j} \phi_{ij} x^i y^j,$$

$$t(x, y) = nx + ly + \sum_{i,j} \psi_{ij} x^i y^j.$$

Let $f = 1 + k^2 + n^2$ and $g = 1 + m^2 + l^2$. Then by substituting the Taylor expansions for $z(x, y)$, and $t(x, y)$ into (0.1), we obtain

$$\begin{aligned} &\phi_{20} = Af, \qquad \phi_{02} = Ag, \\ &\psi_{20} = Bf, \qquad \psi_{02} = Bg, \end{aligned} \tag{0.2}$$

$$\begin{aligned} &\phi_{30} = 2A(kA + nB)f, \qquad \phi_{03} = 2A(mA + lB)g, \\ &\psi_{30} = 2B(kA + nB)f, \qquad \psi_{03} = 2B(mA + lB)g, \end{aligned} \tag{0.3}$$

$$\begin{aligned} &\phi_{40} = A\left(A^2 + B^2\right)f^2 + 4A(kA + nB)^2 f, \\ &\psi_{40} = B\left(A^2 + B^2\right)f^2 + 4B(kA + nB)^2 f, \\ &\phi_{04} = A\left(A^2 + B^2\right)g^2 + 4A(mA + lB)^2 g, \\ &\psi_{04} = B\left(A^2 + B^2\right)g^2 + 4B(mA + lB)^2 g \end{aligned} \tag{0.4}$$

with ϕ_{i0}, ψ_{i0} and ϕ_{0j}, ψ_{0j} are polynomials of degrees at most $2i - 1$ and $2j - 1$ respectively in the variables k, m, n, l (Proposition 3), and A, B are, at the outset, just rational functions in k, m, n, l. We want to show that A and B are polynomials of degree 1. To this end, it suffices to work with those ϕ and ψ functions having the same indices.

Multiplying the first pair of (0.3) by f yields

$$f\phi_{30} = 2Af(kA+nB)f = \phi_{20}(k\phi_{20}+n\psi_{20}),$$
$$f\psi_{30} = 2Bf(kA+nB)f = \psi_{20}(k\phi_{20}+n\psi_{20}). \tag{0.5}$$

According to Proposition 3, the functions (ϕ_{20},ψ_{20}), (ϕ_{30},ψ_{30}) and (ϕ_{40},ψ_{40}) are polynomials of degree at most 3, 5, and 7, respectively, in the variables k,m,n,l. Eqs. (0.5) are satisfied for all values of k, m, n, l corresponding to spheres in general position in the pencil having enough spheres. Since 4-variable polynomials of degree 5 which coincide at the corresponding enough values of k, m, n, l, coincide identically, we see that the equations are in fact identities. These identities imply either: f divides $(k\phi_{20}+n\psi_{20})$, or: f divides both ϕ_{20} and ψ_{20}. In the latter case Eqs. (0.2) show that A and B are polynomials of degree 1. So we may assume that $fh = k\phi_{20}+n\psi_{20}$ for some polynomial h.

Now

$$\begin{aligned} fh &= k\phi_{20}+n\psi_{20} \\ &= kAf + nBf \quad \text{(by Eq. (0.2))} \\ &= (kA+nB)f. \end{aligned}$$

Thus $kA+nB = h$ is a polynomial.

Multiplying Eqs. (0.4) by f yields

$$\begin{aligned} f\phi_{40} &= (Af)((Af)^2+(Bf)^2+4(Af)(kA+nB)^2 f \\ &= \phi_{20}\big(\phi_{20}^2+\psi_{20}^2\big)+4\phi_{20}h^2 f. \end{aligned}$$

Similarly $f\psi_{40} = \psi_{20}(\phi_{20}^2+\psi_{20}^2)+4\psi_{20}h^2 f$.

By the same reasoning as before, these equations are again identities. This shows that either: f divides $(\phi_{20}^2+\psi_{20}^2)$, or: f divides both ϕ_{20} and ψ_{20}. In the second case, as before, we are done, so we may assume that $f|(\phi_{20}^2+\psi_{20}^2)$.

Eq. (0.3) gives

$$\begin{aligned} n\phi_{30}+k\psi_{30} &= 2nA(kA+nB)f + 2kB(kA+nB)f \\ &= 2f(nA+kB)(kA+nB) \\ &= 2f\big(kn(A^2+B^2)+2(k^2+n^2)AB\big), \end{aligned}$$

so $f(n\phi_{30}+k\psi_{30}) = 2kn(\phi_{20}^2+\psi_{20}^2)+2(k^2+n^2)\phi_{20}\psi_{20}$.

Since we already know that $f|(\phi_{20}^2+\psi_{20}^2)$, it follows that $f|\phi_{20}\psi_{20}$. Thus we also have $f|(\phi_{20}\pm\psi_{20})^2$ and thus $f|\phi_{20}\pm\psi_{20}$ and finally $f|\phi_{20}$ and $f|\psi_{20}$.

Since these polynomials are polynomials of degree 3 in k, m, n, l, the functions $A(k,m,n,l)$ and $B(k,m,n,l)$ would be polynomials of degree 1 in k,m,n,l, i.e.,

$$A(k,m,n,l) = ak+bm+cn+dl+e,$$
$$B(k,m,n,l) = \acute{a}k+\acute{b}m+\acute{c}n+\acute{d}l+\acute{e}$$

where $a,b,\ldots,\acute{c},\acute{d}$ are all constants.

Similarly to find the symmetric relations, suppose that the image of the map $\gamma(x,y) = (x,y,z(x,y),t(x,y))$ lies on the following two 3-folds

$$F = f_3 - kf_1 - mf_2 = 0,$$
$$G = f_4 - nf_1 - lf_2 = 0,$$

where $f_i(x,y,z,t)$, $i = 1,\ldots,4$, are the component functions of the rectifying diffeomorphism, and $F\circ\gamma = G\circ\gamma = 0$. By partial differentiation we get

$$\langle dF(\gamma),\gamma_x\rangle = 0,$$
$$\langle dG(\gamma),\gamma_x\rangle = 0.$$

Next by second derivative we obtain

$$\langle d^2F(\gamma)\gamma_x,\gamma_x\rangle + \langle \partial_x F(\gamma),\gamma_{xx}\rangle = 0,$$
$$\langle d^2G(\gamma)\gamma_y,\gamma_y\rangle + \langle \partial_y G(\gamma),\gamma_{yy}\rangle = 0.$$

This expressions hold for any surface containing the sphere under consideration i.e., the image of the map $\gamma(x)$, in particular for $F = f_3 - kf_1 - mf_2$, $G = f_4 - nf_1 - lf_2$. Since

$$\gamma_x(0) = (1, 0, k, n), \qquad \gamma_y(0) = (0, 1, m, l),$$
$$dF\big(\gamma(0)\big) = (-k, -m, 1, 0), \qquad dG\big(\gamma(0)\big) = (-n, -l, 0, 1),$$
$$d^2F\big(\gamma(0)\big) = C - kM - mN, \qquad d^2G = \big(\gamma(0)\big) = D - nM - lN,$$

where M, N, C, and D, are 4×4 symmetric matrices, we get

$$z_{xx}(0) = \big\langle (C + kM + mN)v, v\big\rangle, \qquad t_{yy}(0) = \big\langle (D + nM + lN)w, w\big\rangle$$

where, $v = (1, 0, k, n)$ and $w = (0, 1, m, l)$.

Substituting the values of $z_{xx}(0)$ and $t_{xx}(0)$ into (0.2) and comparing the coefficients of both sides we get

$$a = \acute{c}, \quad b = \acute{d}, \quad c = d = 0, \quad \acute{a} = \acute{b} = 0.$$

These relations imply that

$$A = ak + bm + e, \qquad B = an + bl + \acute{e}.$$

We see that the equations of the pencil reduce to the following form:

$$\begin{cases} S_1 + kS_3 + mS_4 = 0, \\ S_2 + nS_3 + lS_4 = 0 \end{cases} \tag{0.6}$$

where,

$$S_1 : -z + e(x^2 + y^2 + z^2 + t^2) = 0,$$
$$S_2 : -t + \acute{e}(x^2 + y^2 + z^2 + t^2) = 0,$$
$$S_3 : x + a(x^2 + y^2 + z^2 + t^2) = 0,$$
$$S_4 : y + b(x^2 + y^2 + z^2 + t^2) = 0,$$

are the equations for certain non-tangent spheres passing through the origin having tangent spaces in general position. We denote by Q the second point of intersection of the spheres $S_1 = 0$, $S_2 = 0$, S_3 and $S_4 = 0$. All the spheres in the pencil pass through Q. In order to rectify such a pencil, it suffices to take the point Q to infinity via a conformal transformation. The proof for $n = 4$ is now complete.

Next suppose that $n = n_0 + 4$, where n_0 is a non-negative integer. As we did before, the proof for this part is given by induction on n for a pencil of spheres of co-dimension 2 containing at least $n + 50$ spheres in $\mathbb{R}^n$. For $n = 5$, this is just the rectification theorem for a bundle of circles in $\mathbb{R}^3$ [1]. We fix one sphere and call it S_0. Then we take the intersections of S_0 with the remaining spheres from the pencil to get a rectifiable bundle of at least 54 circles in general position in $\mathbb{R}^3$. Let ψ be the stereographic projection which sends S_0 into the space $\mathbb{R}^3$. If Φ is the desired rectifying map, then the composition map $\Phi \circ \psi^{-1}$ is a map which sends the bundle of circles in the space $\mathbb{R}^3$ (i.e., the image of circles on the fixed sphere S_0 under stereographic projection) into straight lines. Since this bundle contains at least 54 circles, by the rectification problem for the bundle of circles in 3-dimensional space, it should pass through a second point p distinct from the point $\psi(0)$. It follows that all the circles on the S_0 and consequently all the spheres in the main pencil of spheres pass through $\psi^{-1}(p)$ which is distinct from the origin. Now suppose that the statement holds for $n = k$. We will prove that it holds for $n = k + 1$. So suppose that a pencil of $(k-1)$-dimensional spheres passing through the origin and containing at least $k + 1 + 50$ spheres in general position in R^{k+1} is locally rectifiable, then we fix one sphere and call it S_1. Then take the intersections of S_1 with the remaining spheres from the main pencil. Clearly the new pencil will be a pencil of $(k-2)$-dimensional spheres which contains at least $(k + 50)$ spheres in general position in R^k. By induction, this new pencil of spheres pass through another point distinct from the origin. It follows that the main pencil of spheres also pass through the same point. The proof is complete. □

Acknowledgements

The author would like to extend his thanks to the anonymous referee for his careful reading and many valuable suggestions that have greatly improved this paper. He is also grateful to Askold Khovanskii for fruitful discussions on this topic and his encouragement for publication.

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